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LETTER TO THE EDITOR

A new class of quasi-exactly solvable potentials with a position-dependent mass

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Abstract
A new class of quasi-exactly solvable potentials with a variable mass in the Schrödinger equation is presented. We have derived a general expression for the potentials, including Natanzon confluent potentials. The general solution of the Schrödinger equation is determined and the eigenstates are expressed in terms of the orthogonal polynomials.

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In recent years, physical systems with a position-dependent mass [1–3] and quasi-exactly solvable (QES) potentials [4] have been the focus of interest. In quantum mechanics there exist potentials for which it is possible to find a finite number of eigenvalues and associated eigenfunctions exactly, and in a closed form. These systems are said to be quasi-exactly solvable. The effective mass models have been used to describe electronic properties of semiconductors, liquid crystals and various other physical systems [5]. In this letter, we suggest a method to obtain a general solution of the Schrödinger equation with a position-dependent mass.

We start with a general Hermitian effective mass Hamiltonian which is proposed by von Roos [6],

$$H = \tfrac{1}{4}(m^{\alpha}(x)\mathbf{p}m^{\beta}(x)\mathbf{p}m^{\gamma}(x) + m^{\gamma}(x)\mathbf{p}m^{\beta}(x)\mathbf{p}m^{\alpha}(x)) + V(x) \tag{1}$$

with the constraint over the parameters: $\alpha + \beta + \gamma = -1$. Depending on the choice of the parameters the Hamiltonian (1) can be expressed in different forms [2]. However, we shall keep the general form of the Hamiltonian. Using the differential operator equivalence of momentum operator $\mathbf{p} = -\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}x}$, it is easy to show that the Hamiltonian (1) can be written as

$$-\frac{1}{2m(x)}\frac{\mathrm{d}^2\psi(x)}{\mathrm{d}x^2} + \frac{m'(x)}{2m^2(x)}\frac{\mathrm{d}\psi(x)}{\mathrm{d}x} + (V(x) - E)\psi(x) + [(1+\beta)m(x)m''(x) \\ - 2(\beta + 1 + \alpha(\alpha+\beta+1)m'^2(x)]\frac{\psi(x)}{4m^3(x)} = 0 \tag{2}$$

where E is the eigenvalue of the Hamiltonian (1). Our task is now to obtain a general expression for the potential $V(x)$ such that the Schrödinger equation can be solved quasi-exactly. Without loss of generality, consider the following QES second-order differential equation [4]:

$$z\frac{\mathrm{d}^2\,\mathrm{R}(z)}{\mathrm{d}z^2} + \left(\ell + \frac{3}{2} + z(b - qz)\right)\frac{\mathrm{d}\,\mathrm{R}(z)}{\mathrm{d}z} + (-\varepsilon + 2jqz)\,\mathrm{R}(z) = 0 \tag{3}$$

where ℓ, b, q and ε are constants and j takes integer and half-integer values. The function $\mathrm{R}(z)$ is a polynomial of degree $2j$. The differential equation can be obtained by introducing the following linear and bilinear combinations of the generators of the $sl(2, R)$ Lie algebra:

$$[J_- J_0 + (\ell + j + 1/2)J_- + qJ_+ + bJ_0 + (-\varepsilon + jb)]\,\mathrm{R}(z) = 0 \tag{4}$$

which is quasi-exactly solvable (QES) [4]. The differential realizations of the generators of the algebra are given by [4]

$$J_- = \frac{\mathrm{d}}{\mathrm{d}z} \qquad J_0 = z\frac{\mathrm{d}}{\mathrm{d}z} - j \qquad J_+ = -z^2\frac{\mathrm{d}}{\mathrm{d}z} + 2jz. \tag{5}$$

The function $\mathrm{R}(z)$ forms a basis for $sl(2, R)$ Lie algebra. The solution of the differential equation (3), which was determined in [4], is in the following form:

$$\mathrm{R}_j(z^2) = \sum_{m=0}^{2j} \frac{(2j)!(2\ell+1)!(\ell+m)!}{2m!(2j-m)!(2\ell+1+2m)!} P_m(\varepsilon)(-qz^2)^m. \tag{6}$$

Here, the polynomial $P_m(\varepsilon)$ satisfies the recurrence relation

$$(2j - m)qP_{m+1}(\varepsilon) - (\varepsilon - bm)P_m(\varepsilon) + m(\ell + m + 1/2)P_{m-1}(\varepsilon) = 0 \tag{7}$$

with the initial condition $P_0(\varepsilon) = 1$. The polynomial $P_m(\varepsilon)$ vanishes for $m \geqslant 2j + 1$ and the roots of $P_{2j+1}(\varepsilon) = 0$ correspond to the ε-eigenvalues of the algebraic Hamiltonian (4). It is well known that the differential equation (3) can be transformed into the form of the Schrödinger equation and several quantum-mechanical potentials can be generated. In order to discuss all the potentials related to the differential equation (3), in a unified manner we introduce a variable $z = r(x)$, then equation (3) takes the form

$$\frac{r}{r'^2}\frac{\mathrm{d}^2\,\mathrm{R}(x)}{\mathrm{d}x^2} + \frac{1}{r'}\left[\ell + 3/2 + r(b - qr) - \frac{rr''}{r'^2}\right]\frac{\mathrm{d}\,\mathrm{R}(x)}{\mathrm{d}x} + (-\varepsilon + 2jqr)\,\mathrm{R}(x) = 0. \tag{8}$$

Now let us turn our attention to the effective mass Schrödinger equation (2). In this case, both the Schrödinger equation and the QES differential equation (8) include first-order differential terms. One can easily transform the effective mass Schrödinger equation into the form of (8). It is convenient to express the eigenfunction $\psi(x)$ in the usual form

$$\psi(x) = -\frac{2r}{r'^2}m(x)\,\mathrm{e}^{-\int W(x)\,\mathrm{d}x}\,\mathrm{R}(x). \tag{9}$$

Substituting (9) into (2) and then comparing with (8) we obtain the following expression for the weight function $W(x)$:

$$W(x) = \frac{1}{4}\left(\frac{2m'(x)}{m(x)} - \frac{6r''}{r'} + \frac{(1 - 2\ell - 2br + 2qr^2)r'}{r}\right) \tag{10}$$

and an implicit expression for the potential function as follows

$$\begin{aligned} m(x)\,[V(x) - E] = {} & \frac{(\beta + 1/4 + \alpha(\alpha + \beta + 1))m'^2(x)}{2m^2(x)} - \beta\frac{m''(x)}{4m(x)} \\ & + \frac{3}{8}\left(\frac{r''}{r'}\right)^2 - \frac{r'''}{4r'}\Bigg(b^2 - (2\ell + 8j + 5)q \\ & + \frac{4\varepsilon + b(2\ell + 3)}{r} + \frac{\ell(\ell+1) - 3/4}{r^2} - 2bqr + q^2r^2\Bigg)\frac{r'^2}{8} \end{aligned} \tag{11}$$

where r^i is the ith derivative of r with respect to x.

At this point we first discuss the special form of the above potential. When we choose $q = 0$ the potential is exactly solvable. Under the conditions $q = 0$ and $m(x) = \text{constant}$, the potential leads to the Natanzon confluent potentials [7]. To obtain the quantum-mechanical potentials we have to get $m(x)E$ on the left-hand side, there must be at least one term on the right-hand side from which a constant times $m(x)$ arises [8]. The last term gives a constant if the function $r(x)$ satisfies the relation

$$\sqrt{\lambda_0 + \lambda_1/r(x) + \lambda_2/r^2(x)}\,\frac{\mathrm{d}r}{\mathrm{d}x} = -\sqrt{m(x)} \tag{12}$$

where λ_0, λ_1 and λ_2 are constants. In the following, we consider the potentials associated with the radial sextic oscillator potential, the QES Coulomb potential and the Morse potential.

In order to obtain the radial sextic oscillator family potential we chose $\lambda_0 = \lambda_2 = 0$ and $\lambda_1 = 1/4$, then $r(x) = -u^2 = -[\int \sqrt{m(x)}\,\mathrm{d}x]^2$ and the potential takes the form

$$\begin{aligned} V(x) = {}& \frac{\ell(\ell+1)}{2u^2} + \frac{1}{2}(b^2 - (2\ell + 8j + 5)q)u^2 + bqu^4 + \frac{1}{2}q^2u^6 \\ & + \frac{(\alpha(\alpha+\beta+1) + \beta + 9/16)\,m'^2(x)}{2m^3(x)} - \frac{(1+2\beta)m''(x)}{8m^2(x)}. \end{aligned} \tag{13}$$

This is a family of radial sextic oscillator potentials. We have checked that for the choice $q = 0$ and $m(x) = \left(\frac{a+x^2}{1+x^2}\right)^2$ the potential takes the same form as the potential given in [2], and for $m(x) = cx^2$ the potential corresponds to the potential given by Dutra [3]. The eigenvalue of the Schrödinger equation with the potential given in (7) is given by

$$E = \left(\ell + \frac{3}{2}\right)b + 2\epsilon. \tag{14}$$

The energy parameter ε is obtained from the recurrence relation (7).

For the cases $\lambda_1 = \lambda_2 = 0$ and $\lambda_0 = 1/4$ the function $r(x) = -2u$ and the potential takes the form

$$\begin{aligned} V(x) = {}& \frac{\ell(\ell+1) - 3/4}{8u^2} - \frac{4\varepsilon + (2\ell+3)b}{4u} + 2bqu + 2q^2u^2 \\ & + \frac{(\alpha(\alpha+\beta+1) + \beta + 9/16)\,m'^2(x)}{2m^3(x)} - \frac{(1+2\beta)m''(x)}{8m^2(x)}. \end{aligned} \tag{15}$$

This potential represents a family of QES Coulomb potentials. In order to obtain the standard form of the potential one should redefine the parameters. The eigenvalues of the potential are given by

$$E = -\tfrac{1}{2}((2\ell + 8j + 5)q - b^2). \tag{16}$$

For the last example we choose $\lambda_0 = \lambda_1 = 0$ and $\lambda_2 = 1$ to obtain a family of QES Morse potentials. Then $r(x) = \mathrm{e}^{-u}$ and the potential takes the form

$$\begin{aligned} V(x) = {}& \frac{1}{2}\,(\varepsilon + (\ell/2 + 3/4)b)\,\mathrm{e}^{-u} + \frac{1}{2}(b^2/4 - (\ell/2 + j + 5/4)q)\,\mathrm{e}^{-2u} - \frac{bq}{4}\,\mathrm{e}^{-3u} \\ & + \frac{q^2}{8}\,\mathrm{e}^{-4u} + \frac{(\alpha(\alpha+\beta+1) + \beta + 9/16)\,m'^2(x)}{2m^3(x)} - \frac{(1+2\beta)m''(x)}{8m^2(x)}. \end{aligned} \tag{17}$$

The standard form of the Morse potential can be obtained by reordering the parameters. The corresponding eigenvalue is given by

$$E = -\tfrac{1}{8}\,(\ell(\ell+1) + 1/4)\,. \tag{18}$$

We have constructed a class of QES potentials for the generalized effective mass Hamiltonian without any restriction on the parameters α and β. We have shown that one can obtain a family of potentials, related to the sextic oscillator, QES Coulomb and QES Morse potentials. The method discussed here can be used for obtaining other classes of potentials which can be related to the hypergeometric Natanzon class of potentials.

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